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Generalizations of a parameterized Jordan-type inequality, Janous's inequality and Tsintsifas's inequality

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ABSTRACT

In this work, we first prove a generalized version of a parameterized Jordan-type inequality. We then use it to prove the generalized versions of Janous's inequality and Tsintsifas's inequality which reduce to two inequalities conjectured by Janous and Tsintsifas as special cases.

1. Introduction

The celebrated Jordan's inequality is stated as Theorem A (see Mitrinović and Vasić [1, p.33]).

Theorem A. *Let* $0 < x \le \frac{\pi}{2}$. *Then*

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1. \tag{1}$$

Jordan's inequality (1) and its modified versions have many important applications in calculus, trigonometry and the theory of limits. In recent years, this classical inequality has received considerable attention, different proofs, various generalizations, improvements and analogues have been presented by many authors (see [2–17] and references therein).

Recently, Janous [18] and Tsintsifas [19] proposed the following inequalities involving the ratio $(\sin x)/x$ as conjectures:

Conjecture 1 (*See Janous [18]*). *Suppose* $A > 0, B > 0, C > 0$ *and* $A + B + C = \pi$. *Then the following inequality holds true:*

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi}. \tag{2}$$

Conjecture 2 (*See Tsintsifas [19]*). *Suppose* $0 < A < \pi/2, 0 < B < \pi/2, 0 < C < \pi/2$ *and* $A + B + C = \pi$. *Then the following inequality holds true:*

$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}. \tag{3}$$

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Recently, Klamkin [20] and Murty et al. [21] have given affirmative proofs of Conjectures 1 and 2 respectively. However, it is of interest to consider more general inequalities of Janous-type and of Tsintsifas-type which may hold under the weak condition $A + B + C = \theta$ where $0 < \theta \leq \pi$. We are inspired by the following parameterized Jordan-type inequality presented in our recent paper [22] to pursue the proposed results.

Theorem B (*See Wu and Debnath [22]*). *Let* $0 < x \leq \theta \leq \frac{\pi}{2}$. *Then*

$$\frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x-\theta) \leq \frac{\sin x}{x} \leq \frac{\sin\theta}{\theta} + \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta). \tag{4}$$

It is important to note that the inequality (4) cannot be applied in the improvement of inequalities (2) and (3) because the condition of parameter is $0 < \theta \leq \pi/2$ rather than $0 < \theta \leq \pi$ which is what we would need for the inequality to be applied. Therefore, a further study on the Jordan-type inequality (4) is needed.

In this work, we shall generalize the inequality (4). Under certain weak constraint conditions, we prove that inequality (4) is still valid. In Section 3, the result will be used to generalize Janous's inequality (2) and improve Tsintsifas's inequality (3).

2. Generalization of Jordan-type inequality

In the generalization of Jordan-type inequality (4), we relax the constraint conditions of parameter in (4), the main result is the following theorem:

Theorem 1. *If* $0 < x \leq \theta \leq \pi$, *then*

$$\frac{\sin x}{x} \geq \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x-\theta). \tag{5}$$

If $0 < x \leq \pi$ *and* $0 < \theta \leq \frac{\pi}{2}$, *then*

$$\frac{\sin x}{x} \leq \frac{\sin\theta}{\theta} + \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta). \tag{6}$$

Furthermore, the equalities hold in (5) or (6) if and only if $x = \theta$.

Proof. We introduce a function

$$f : (0, \pi] \longrightarrow \mathbb{R}$$

by

$$f(x) = \frac{1}{x}\left(\frac{\sin x}{x} - 1\right).$$

Differentiating $f(x)$ with respect to x gives

$$f'(x) = \frac{x\cos x + x - 2\sin x}{x^3}.$$

By means of a simple calculation and the well-known inequality $\tan\alpha > \alpha$ $(0 < \alpha < \pi/2)$, we find that for $0 < x < \pi$,

$$f'(x) = \frac{x\cos x + x - 2\sin x}{x^3} = \frac{4\cos^2\frac{x}{2}\left(\frac{x}{2} - \tan\frac{x}{2}\right)}{x^3} < 0.$$

This means that $f(x)$ is strictly decreasing on $(0, \pi)$. Thus, we deduce that for $0 < x < \theta \leq \pi$,

$$\frac{1}{x}\left(\frac{\sin x}{x} - 1\right) > \frac{1}{\theta}\left(\frac{\sin\theta}{\theta} - 1\right),$$

which can be simplified to the desired inequality:

$$\frac{\sin x}{x} > \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x-\theta). \tag{7}$$

In addition, it is easy to see that the equality occurs in (5) if and only if $x = \theta$. The inequality (5) is thus proved.

We next define a function

$$g : (0, \pi] \longrightarrow \mathbb{R}$$

by

$$g(x) = \frac{\sin x}{x} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta) - \frac{\sin\theta}{\theta}.$$

Differentiating $g(x)$ with respect to x gives

$$g'(x) = \frac{x\cos x - \sin x}{x^2} - \frac{\theta\cos\theta - \sin\theta}{\theta^2},$$

$$g''(x) = \frac{2\sin x - 2x\cos x - x^2\sin x}{x^3}.$$

Let $\varphi(x) = 2\sin x - 2x\cos x - x^2\sin x$, then $\varphi'(x) = -x^2\cos x$. In view of the fact that $\varphi'(x) < 0$ for $0 < x < \frac{\pi}{2}$ and $\varphi'(x) > 0$ for $\frac{\pi}{2} < x < \pi$, we conclude that $\varphi(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

Further, we deduce from the observation $\varphi\left(\frac{\pi}{2}\right) < 0$ and $\varphi(\pi) > 0$ that there is a point $x_1 \in \left(\frac{\pi}{2}, \pi\right)$ such that $\varphi(x) < 0$ for $0 < x < x_1$ and $\varphi(x) > 0$ for $x_1 < x < \pi$. We then conclude that $g'(x)$ is strictly decreasing on $(0, x_1)$ and strictly increasing on (x_1, π).

In addition, it follows from straightforward computation that

$$g'(0+) = \frac{\cos\theta(\tan\theta - \theta)}{\theta^2} > 0, \qquad g'(\theta) = 0 \quad (0 < \theta \le \pi/2 < x_1), \tag{8}$$

which implies that

$$g'(x_1) < g'\left(\frac{\pi}{2}\right) \le 0. \tag{9}$$

To prove the required inequality (6), we consider the following two cases.

Case (I): When $g'(\pi) \le 0$.

By relations (8), (9), $g'(\pi) \le 0$ and the monotonicity of $g'(x)$, it follows that $g'(x) > 0$ for $0 < x < \theta$ and $g'(x) < 0$ for $\theta < x < \pi$. We hence conclude that $g(x)$ is strictly increasing on $(0, \theta)$ and strictly decreasing on (θ, π).

Therefore, we have

$$g(x) \le g(\theta) = 0 \quad \text{for } 0 < x \le \pi, \tag{10}$$

which leads us to the desired inequality (6).

Case (II): When $g'(\pi) > 0$.

Since $g'(x_1) < 0$, $g'(\pi) > 0$ and $g'(x)$ is strictly increasing on (x_1, π), there exists a point $x_2 \in (x_1, \pi)$ such that $g'(x) < 0$ for $x_1 < x < x_2$ and $g'(x) > 0$ for $x_2 < x < \pi$. On the other hand, we observe via relations (8) and (9) that $g'(x) > 0$ for $0 < x < \theta$ and $g'(x) < 0$ for $\theta < x < x_1$.

Consequently, $g(x)$ is strictly increasing on $(0, \theta)$, strictly decreasing on (θ, x_2) and strictly increasing on (x_2, π).

Now, from $g(\theta) = 0$ and the following inequality holds:

$$\begin{aligned} g(\pi) &= -\frac{\theta\cos\theta - \sin\theta}{\theta^2}(\pi - \theta) - \frac{\sin\theta}{\theta} \\ &= \frac{\pi}{2}\left(-\frac{4}{\pi^2} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}\right) + \left(\frac{2}{\pi} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}\left(\frac{\pi}{2} - \theta\right) - \frac{\sin\theta}{\theta}\right) \\ &= \frac{\pi}{2}g'\left(\frac{\pi}{2}\right) + g\left(\frac{\pi}{2}\right) \le \frac{\pi}{2}g'(\theta) + g(\theta) = 0. \end{aligned}$$

Finally, we deduce that

$$g(x) \le 0 \quad \text{for } 0 < x \le \pi, \tag{11}$$

which implies the desired inequality (6). This completes the proof of Theorem 1. □

Remark 1. It is obvious that the Jordan-type inequality (4) will follow as a special case of Theorem 1 when $0 < x \le \theta \le \pi/2$.

3. Some applications

As applications of Theorem 1, Janous's inequality (2) and Tsintsifas's inequality (3) mentioned in Section 1 will be generalized and improved in this section.

Theorem 2. *Let $x_i > 0$ $(i = 1, 2, \ldots, n,\ n \geq 2)$, $x_1 + x_2 + \cdots + x_n = \theta$ and $0 < \theta \leq \pi$. Then*

$$\frac{\sin\theta}{\theta} + n - 1 < \sum_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \frac{n^2}{\theta} \sin\frac{\theta}{n}, \tag{12}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n = \theta/n$.

Proof. Since $0 < x_i < \theta \leq \pi$ $(i = 1, 2, \ldots, n)$, we obtain via inequality (5) that

$$\frac{\sin x_i}{x_i} > \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x_i - \theta) \quad (i = 1, 2, \ldots, n). \tag{13}$$

Taking the sum for all inequalities of (13) yields

$$\sum_{i=1}^{n} \frac{\sin x_i}{x_i} > \frac{n\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2} \sum_{i=1}^{n}(x_i - \theta)$$
$$= \frac{\sin\theta}{\theta} + n - 1. \tag{14}$$

The hypothesis of Theorem 2 shows that

$$0 < x_i \leq \pi, \quad 0 < \theta/n \leq \pi/2,\ i = 1, 2, \ldots, n,\ n \geq 2.$$

Hence, from inequality (6) we deduce that

$$\frac{\sin x_i}{x_i} \leq \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} + \frac{\frac{\theta}{n}\cos\frac{\theta}{n} - \sin\frac{\theta}{n}}{\left(\frac{\theta}{n}\right)^2}\left(x_i - \frac{\theta}{n}\right) \quad (i = 1, 2, \ldots, n), \tag{15}$$

where the equality holds if and only if $x = \theta/n$.

Taking the sum for all inequalities of (15), we get

$$\sum_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \frac{n\sin\frac{\theta}{n}}{\frac{\theta}{n}} + \frac{\frac{\theta}{n}\cos\frac{\theta}{n} - \sin\frac{\theta}{n}}{\left(\frac{\theta}{n}\right)^2} \sum_{i=1}^{n}\left(x_i - \frac{\theta}{n}\right)$$
$$= \frac{n^2}{\theta}\sin\frac{\theta}{n}.$$

This completes the proof of Theorem 2. □

Setting $n = 3$, $x_1 = A$, $x_2 = B$, $x_3 = C$ in Theorem 2, a generalization of Janous's inequality (2) is derived as follows:

Corollary 1. *Let $A > 0$, $B > 0$, $C > 0$, $A + B + C = \theta$ and $0 < \theta \leq \pi$. Then*

$$\frac{\sin\theta}{\theta} + 2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \leq \frac{9}{\theta}\sin\frac{\theta}{3}. \tag{16}$$

It is important to point out that Corollary 1 with the special case $\theta = \pi$ yields Janous's inequality (2).

Theorem 3. *Let $x_i > 0$ $(i = 1, 2, \ldots, n,\ n \geq 2)$, $x_1 + x_2 + \cdots + x_n = \theta$ and $0 < \theta \leq \pi$. Then*

$$\sum_{i=1}^{n} \frac{\sin x_i}{\theta - x_i} > \frac{1}{n-1}\left(\frac{\sin\theta}{\theta}\right) + 1. \tag{17}$$

Proof. Note that $0 < x_i < \theta \leq \pi$ $(i = 1, 2, \ldots, n)$, then it follows from inequality (5) that

$$\frac{\sin x_i}{\theta - x_i} > \left(\frac{\sin\theta}{\theta}\right)\frac{x_i}{\theta - x_i} + \left(\frac{\theta - \sin\theta}{\theta^2}\right)x_i$$
$$= \frac{\sin\theta}{\theta - x_i} + \left(\frac{\theta - \sin\theta}{\theta^2}\right)x_i - \frac{\sin\theta}{\theta} \quad (i = 1, 2, \ldots, n). \tag{18}$$

Summing up (18) for $i = 1, 2, \ldots, n$ yields

$$\sum_{i=1}^{n} \frac{\sin x_i}{\theta - x_i} > \sin\theta\left(\sum_{i=1}^{n}\frac{1}{\theta - x_i}\right) + \left(\frac{\theta - \sin\theta}{\theta^2}\right)\left(\sum_{i=1}^{n} x_i\right) - \frac{n\sin\theta}{\theta}$$
$$= \sin\theta\left(\sum_{i=1}^{n}\frac{1}{\theta - x_i}\right) - \frac{(n+1)\sin\theta}{\theta} + 1. \tag{19}$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\sum_{i=1}^{n} \frac{1}{\theta - x_i} \geq n^2 \left(\sum_{i=1}^{n} (\theta - x_i) \right)^{-1} = \frac{n^2}{(n-1)\,\theta}. \tag{20}$$

Substituting (20) into (19) leads to inequality (17). The proof of Theorem 3 is complete. □

Setting $n = 3, x_1 = A, x_2 = B, x_3 = C$ in Theorem 3, we get the generalized version of Tsintsifas's inequality (3):

Corollary 2. *Let $A > 0, B > 0, C > 0, A + B + C = \theta$ and $0 < \theta \leq \pi$. Then*

$$\frac{\sin A}{\theta - A} + \frac{\sin B}{\theta - B} + \frac{\sin C}{\theta - C} > \frac{\sin\theta}{2\theta} + 1. \tag{21}$$

If we put $\theta = \pi$ in inequality (21), we obtain the following result:

$$\frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} > 1. \tag{22}$$

Inequality (22) partially improves Tsintsifas's inequality (3), since the lower bound of (22) is sharper than that of (3).

Theorem 4. *Let $x_i > 0 \ \ (i = 1, 2, \ldots, n, \ \ n \geq 3), x_1 + x_2 + \cdots + x_n = \theta$ and $0 < \theta \leq \pi$. Then*

$$(n-1)\left(\frac{\sin\theta}{\theta}\right) + 1 < \sum_{i=1}^{n} \frac{\sin(\theta - x_i)}{\theta - x_i} < \left(n^2 - 3n + 1\right)\cos\frac{\theta}{n-1} - \frac{(n-1)\left(n^2 - 4n + 1\right)}{\theta}\sin\frac{\theta}{n-1}. \tag{23}$$

Proof. The hypothesis of Theorem 4 implies that

$$0 < \theta - x_i < \theta \leq \pi, \quad 0 < \theta/(n-1) \leq \pi/2, \ i = 1, 2, \ldots, n, \ n \geq 3.$$

Thus, using inequalities (5) and (6), we obtain that for $i = 1, 2, \ldots, n$,

$$\begin{aligned} \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(\theta - x_i - \theta) &< \frac{\sin(\theta - x_i)}{\theta - x_i} \\ &< \frac{\sin\frac{\theta}{n-1}}{\frac{\theta}{n-1}} + \frac{\frac{\theta}{n-1}\cos\frac{\theta}{n-1} - \sin\frac{\theta}{n-1}}{\left(\frac{\theta}{n-1}\right)^2}\left(\theta - x_i - \frac{\theta}{n-1}\right). \end{aligned} \tag{24}$$

Summing up (24) for $i = 1, 2, \ldots, n$ yields

$$\begin{aligned} \frac{n\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}\sum_{i=1}^{n}(\theta - x_i - \theta) &< \sum_{i=1}^{n}\frac{\sin(\theta - x_i)}{\theta - x_i} \\ &< \frac{n\sin\frac{\theta}{n-1}}{\frac{\theta}{n-1}} + \frac{\frac{\theta}{n-1}\cos\frac{\theta}{n-1} - \sin\frac{\theta}{n-1}}{\left(\frac{\theta}{n-1}\right)^2}\sum_{i=1}^{n}\left(\theta - x_i - \frac{\theta}{n-1}\right). \end{aligned} \tag{25}$$

A simple computation reduces inequality (25) to inequality (23). The proof of Theorem 4 is complete. □

Setting $n = 3, x_1 = A, x_2 = B, x_3 = C$ in Theorem 4 yields the following generalized and sharpened version of Tsintsifas's inequality (3).

Corollary 3. *Let $A > 0, B > 0, C > 0, A + B + C = \theta$ and $0 < \theta \leq \pi$. Then*

$$\frac{2\sin\theta}{\theta} + 1 < \frac{\sin(\theta - A)}{\theta - A} + \frac{\sin(\theta - B)}{\theta - B} + \frac{\sin(\theta - C)}{\theta - C} < \cos\frac{\theta}{2} + \frac{4}{\theta}\sin\frac{\theta}{2}. \tag{26}$$

In particular, putting $\theta = \pi$ in Corollary 3, we obtain:

Corollary 4. *Let $A > 0, B > 0, C > 0$ and $A + B + C = \pi$. Then*

$$1 < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{4}{\pi}. \tag{27}$$

Obviously, inequality (27) is stronger than Tsintsifas's inequality (3), and also, it is more general than Tsintsifas's result.

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